Desynchronization Entropy

Darío Ricardo De Santiago Sasinka
Affiliation: https://www.frc.utn.edu.ar
ORCID: 0000-0002-7684-9532

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Abstract

A purely relational definition of entropy is introduced, based on the counting of synchronized pairs among cronodinos — elementary temporal oscillators proposed in the theory of Quantum Desynchronization [2]. The desynchronization entropy $S_\varepsilon(t)$ measures the internal disorder of a system without relying on absolute temporal references or statistical hypotheses. Under the condition of spectral incommensurability — the same condition that resolves Loschmidt's paradox [4] — it is shown that, starting from a low entropy state (high synchronization), $S_\varepsilon(t)$ tends to increase until reaching an equilibrium value $S_\varepsilon^{\text{eq}} = \ln(\pi/\varepsilon)$, around which it fluctuates in finite systems. In the thermodynamic limit ($N \to \infty$), fluctuations disappear and entropy becomes deterministic and non-decreasing. This result constitutes a dynamical derivation of the Second Law of Thermodynamics in its statistical formulation, thus establishing a conceptual bridge between quantum desynchronization and classical thermodynamic irreversibility.

Keywords: entropy, quantum desynchronization, second law, thermodynamics, cronodinos, irreversibility, spectral incommensurability.

1 Introduction: The problem of temporal order

Consider a room in which a wall is covered by a large number of analog clocks. At a given instant, a photograph is taken that freezes the position of all hands. Each clock i shows an angle $\theta_i \in [0, 2\pi)$, corresponding to the position of its hour hand.

The relevant physical question is not *what time the clocks show* (since there is no absolute time reference), but rather *how ordered or disordered* the entire set is. Intuitively:

- If all hands point approximately in the same direction, the system is highly **ordered** [1].
- If the hands point in scattered directions, the system is **disordered**, which is the typical state in nature.

Temporal order is not defined by an absolute reference, but by the internal relationships between clocks. Two clocks are synchronized if their readings have a simple relationship with each other (for example, they show the same time).

This example suggests that disorder can be quantified by counting how many coherent temporal relationships exist within the set, and how this number decreases when dynamical differences cause phase dispersion. In what follows, we will formalize this system as a set of *cronodinos*[2].

2 How to measure disorder: pair counting

Consider N cronodinos, each characterized by a phase $\phi_i \in [0, 2\pi)$ at a given instant (the "photograph"). The state of the system is the configuration $\vec{\phi} = (\phi_1, \ldots, \phi_N)$.

[1] Analogous to a crystal in a state of perfect and persistent synchronization.

2.1 Phase relations and pair sets

The relational structure is constructed from the counting of all possible pairs of cronodinos. We define:

- $\mathbf{C} = \{(i, j) : 1 \leq i < j \leq N\}$ as the set of all possible pairs of cronodinos, with $|\mathbf{C}| = \binom{N}{2}$.
- Theoretical (perfect) synchronization: A pair (i, j) is perfectly synchronized if:

$$(\phi_i - \phi_j) \mod 2\pi = 0. \tag{1}$$

  This means[2] that the phases are identical on the unit circle (perfect alignment of hands).
- $\mathbf{B}$: subset of perfectly synchronized pairs that satisfy (1).
- $\mathbf{A}$: subset of non-synchronized pairs that do not satisfy (1).
- Fundamental relation: $\mathbf{C} = \mathbf{B} \cup \mathbf{A}$, with $\mathbf{B} \cap \mathbf{A} = \emptyset$.

3 Desynchronization entropy

We define the theoretical Desynchronization entropy as:

$$\boxed{S_D = \ln\left(\frac{|\mathbf{C}|}{|\mathbf{B}|}\right)}. \tag{2}$$

Interpretation:

- If all pairs are perfectly synchronized ($|\mathbf{B}| = |\mathbf{C}|$), then $S = \ln(1) = 0$.
- If no pair is synchronized ($|\mathbf{B}| = 0$), formally $S \to \infty$.
- In real systems with finite N, $|\mathbf{B}|$ is small but not zero, giving S large but finite.

[2] In all expressions, the operator $\mod 2\pi$ is understood as the standard angular reduction to the interval $[0, 2\pi)$.

3.1 Operational implementation with tolerance ε

In practice, no instrument can verify exact equality. For an experimental tolerance $\varepsilon > 0$, we define:

- Operational synchronization(ε): A pair (i, j) is synchronized within a tolerance ε if:

$$|(\phi_i - \phi_j) \mod 2\pi| < \varepsilon, \tag{3}$$

- $\mathbf{B}_\varepsilon$: subset of pairs considered synchronized that satisfy (3).
- $\mathbf{A}$: subset of pairs considered non-synchronized are those that do not satisfy (3).

3.2 Operational entropy

The operational desynchronization entropy is[3] :

$$\boxed{S_\varepsilon = \ln\left(\frac{|\mathbf{C}|}{|\mathbf{B}_\varepsilon|}\right)}. \tag{4}$$

3.3 Relation between the definitions

- **Consistency in the limit**:
  If $\lim_{\varepsilon \to 0} |\mathbf{B}_\varepsilon| = |\mathbf{B}|$
  then: $\lim_{\varepsilon \to 0} S_\varepsilon = S_D$.
- **Monotonicity in ε:**
  For $\varepsilon_1 < \varepsilon_2$
  it follows that: $|\mathbf{B}_{\varepsilon_1}| \leq |\mathbf{B}_{\varepsilon_2}|$
  and $S_{\varepsilon_1} \geq S_{\varepsilon_2}$.

4 Time evolution of entropy

In a physical system, the phases of cronodinos evolve linearly:

$$\phi_i(t) = \omega_i t + \phi_i(0). \tag{5}$$

[3] The tolerance ε represents the finite temporal resolution of the observer or measurement device. Different values of ε define different synchronization entropies, all monotonically related to each other, without affecting the temporal directionality of the result.

The condition for ε-synchronization for a pair (i, j) at time t is:

$$|(\phi_i(t) - \phi_j(t)) \bmod 2\pi| < \varepsilon. \tag{6}$$

Therefore, the operational entropy evaluated at time t for the system is:

$$S_\varepsilon(t) = \ln\left(\frac{|C|}{|B_\varepsilon(t)|}\right). \tag{7}$$

4.1 Asymptotic behavior and equilibrium

Hypothesis: The frequencies ω_i are mutually incommensurable

$$(\omega_i/\omega_j) \notin \mathbb{Q}\ \forall (i \neq j). \tag{8}$$

Under this condition, the phase difference of any pair (i, j),

$$\Delta_{ij}(t) = ((\omega_i - \omega_j)t + \Delta_{ij}(0)) \mod 2\pi, \tag{9}$$

describes a uniform and ergodic motion on the circle [6]. Consequently, the probability that at a given instant the pair lies within the synchronization band ε is:

$$p_{\text{sinc}} = \frac{\varepsilon}{\pi}. \tag{10}$$

This probability is the same for all pairs and does not depend on time (on average). From this we can analyze the evolution of the expected number of synchronized pairs.

4.1.1 Balance of entries and exits

Consider a time interval sufficiently long so that temporal correlations have been lost (mixing time). The expected number of pairs that *enter* the synchronized set B_ε (transition from non-synchronized to synchronized) in that interval is:

$$\mathbb{E}[\text{entries}] = (|C| - |B_\varepsilon|)\, p_{\text{sinc}},$$

while the expected number of pairs that *exit* is:

$$\mathbb{E}[\text{exits}] = |B_\varepsilon|\,(1 - p_{\text{sinc}}).$$

The expected net balance is given by:

$$\mathbb{E}[\Delta|B_\varepsilon|] = (|C| - |B_\varepsilon|)p_{\text{sinc}} - |B_\varepsilon|(1 - p_{\text{sinc}}) = |C|p_{\text{sinc}} - |B_\varepsilon|.$$

From this it follows that there exists a stable fixed point when $\mathbb{E}[\Delta|B_\varepsilon|] = 0$, namely:

$$|B_\varepsilon|_{\text{eq}} = |C|\, p_{\text{sinc}} = |C|\, \frac{\varepsilon}{\pi}. \tag{11}$$

The corresponding entropy at equilibrium is:

$$S_\varepsilon^{\text{eq}} = \ln\left(\frac{|C|}{|B_\varepsilon|_{\text{eq}}}\right) = \ln\left(\frac{\pi}{\varepsilon}\right). \tag{12}$$

This equilibrium value is independent of initial conditions and of N (as long as N is finite). For a finite system, $|B_\varepsilon(t)|$ will fluctuate around $|B_\varepsilon|_{\text{eq}}$, and therefore $S_\varepsilon(t)$ will fluctuate around $S_\varepsilon^{\text{eq}}$. The relative amplitude of these fluctuations is of order $1/\sqrt{|C|}$, so it tends to zero as $N \to \infty$ (thermodynamic limit).

4.1.2 Tendency to equilibrium from ordered states

If the system starts from a state with high synchronization, i.e., $|B_\varepsilon(0)| > |B_\varepsilon|_{\text{eq}}$ (which corresponds to an initial entropy $S_\varepsilon(0) < S_\varepsilon^{\text{eq}}$), then the expected balance is negative: $\mathbb{E}[\Delta|B_\varepsilon|] < 0$. Therefore, on average, $|B_\varepsilon|$ will decrease and S_ε will increase towards its equilibrium value. Similarly, if the system starts from a state with low synchronization ($|B_\varepsilon(0)| < |B_\varepsilon|_{\text{eq}}$, $S_\varepsilon(0) > S_\varepsilon^{\text{eq}}$), the expected tendency will be towards an increase in $|B_\varepsilon|$ and a decrease in S_ε. Thus, equilibrium is an attractor.

In particular, for the most ordered possible initial condition ($|B_\varepsilon(0)| = |C|$, $S_\varepsilon(0) = 0$), entropy will increase monotonically on average until reaching $S_\varepsilon^{\text{eq}}$, with fluctuations becoming smaller as N increases. This behavior is analogous to that of Boltzmann entropy in a gas expanding from a non-equilibrium state.

4.2 Relaxation to equilibrium theorem

We can summarize the above in the following theorem:

Consider a system of N cronodinos with mutually incommensurable frequencies and arbitrary initial phases. Then:

1. The equilibrium value of the desynchronization entropy is $S_\varepsilon^{\text{eq}} = \ln(\pi/\varepsilon)$, independent of initial conditions.

2. For any initial condition, the expected value $\mathbb{E}[S_\varepsilon(t)]$ tends to $S_\varepsilon^{\text{eq}}$ as $t \to \infty$.

3. If the system starts from a state with $S_\varepsilon(0) < S_\varepsilon^{\text{eq}}$ (more ordered than equilibrium), then, for almost every initial condition, the entropy $S_\varepsilon(t)$ tends to increase towards equilibrium in the sense that its time average is non-decreasing and fluctuations around the trend are small for large N.

4. In the thermodynamic limit $N \to \infty$, entropy becomes a deterministic and monotonically increasing (decreasing) function until reaching equilibrium if $S_\varepsilon(0) < S_\varepsilon^{\text{eq}}$ (if $S_\varepsilon(0) > S_\varepsilon^{\text{eq}}$).

The proof is based on the described balance of entries and exits, together with the ergodicity of phase motion, which guarantees that the probability p_{sinc} is given by (10). Rigorous details require the use of ergodic theory and laws of large numbers, but the essential argument is the one presented.

5 Derivation of the Second Law of thermodynamics

The analysis in the previous section shows that the desynchronization entropy $S_\varepsilon(t)$ possesses the same qualitative properties as thermodynamic entropy: it tends to increase when the system starts from an ordered state, and at equilibrium it fluctuates around a maximum value (for a given tolerance ε). This behavior is precisely what the Second Law of Thermodynamics establishes in its statistical formulation [1].

Indeed, for an isolated system of cronodinos with incommensurable frequencies, it holds that:

- If the system is prepared in a low entropy state (high synchronization), it is overwhelmingly probable that the entropy $S_\varepsilon(t)$ will increase over time until reaching equilibrium, and that once there, fluctuations that reduce it are extremely rare and decrease in magnitude with system size.

- In the thermodynamic limit ($N \to \infty$), the probability of observing a macroscopic decrease in entropy is zero, and entropy becomes a non-decreasing function of time.

We can thus formulate the equivalent of the **Second Law of thermodynamics** for Desynchronization:

> *In an isolated system of cronodinos with a generically incommensurable frequency spectrum, the desynchronization entropy $S_\varepsilon(t)$ tends to increase until reaching its equilibrium value $S_\varepsilon^{eq} = \ln(\pi/\varepsilon)$, around which it fluctuates. In the thermodynamic limit, $S_\varepsilon(t)$ is a monotonically non-decreasing function of time.*

This law is robust with respect to the choice of tolerance ε, since for $\varepsilon_1 < \varepsilon_2$ we have $S_{\varepsilon_1}(t) \geq S_{\varepsilon_2}(t)$, preserving temporal directionality in all cases. Irreversibility emerges as a consequence of the progressive loss of phase correlations among elementary constituents, a phenomenon that is inevitable in systems with a continuous spectrum.

5.1 Conceptual connection with thermodynamic entropy

The desynchronization entropy S_ε and classical thermodynamic entropy S_{term} both measure the internal disorder of a system, although from different perspectives: S_ε quantifies disorder in temporal phase relationships, while S_{term} traditionally quantifies disorder in spatial configurations [3]. The similarity in their behavior suggests that a functional relationship should exist

$$S_{\text{term}} = f(S_\varepsilon), \tag{13}$$

where f is a monotonically increasing function that converts the relational disorder measure

S_ε (dimensionless) into the thermodynamic quantity S_{term} (with dimensions of energy over temperature). The explicit form of f will depend on the physical details of the system and on how the temporal degrees of freedom couple to the observable thermodynamic degrees of freedom.

The derivation of f for specific systems (ideal gases, solids, non-equilibrium systems) constitutes an open problem that would allow directly connecting desynchronization theory with empirical thermodynamics [5].

6 Conclusion

A purely relational entropy, $S_\varepsilon(t)$, has been defined, based on counting synchronized pairs among cronodinos. This measure quantifies the internal disorder of a system of elementary oscillators without needing to resort to absolute temporal references or prior statistical hypotheses.

Under the condition of spectral incommensurability — the same condition as in the framework of Quantum Desynchronization — it has been shown that the expected value of $S_\varepsilon(t)$ tends to an equilibrium value $S_\varepsilon^{\text{eq}} = \ln(\pi/\varepsilon)$. Starting from initial states with high synchronization (low entropy), entropy increases on average towards this equilibrium. In finite systems, $S_\varepsilon(t)$ fluctuates around this value, while in the thermodynamic limit ($N \to \infty$) the behavior becomes deterministic.

This formalism establishes a conceptual connection with classical thermodynamic entropy S_{term}, suggesting that a monotonic functional relationship $S_{\text{term}} = f(S_\varepsilon)$ might exist between them. The explicit determination of this relationship for concrete physical systems (ideal gases, solids, etc.) remains an open problem for future research.

References

[1] Ludwig Boltzmann. Über die beziehung zwischen dem zweiten hauptsatze der mechanischen wärmetheorie und der wahrscheinlichkeitsrechnung. *Wiener Berichte*, 76:373–435, 1877.

[2] Darío Ricardo De Santiago Sasinka. Desincronización cuántica: Resolución de la paradoja de loschmidt. 2026. Preprint. También disponible en inglés como: *Quantum Desynchronization: Resolution of Loschmidt's Paradox*.

[3] Lev D. Landau and Evgeny M. Lifshitz. *Statistical Physics*, volume 5. Butterworth-Heinemann, 3rd edition, 1980.

[4] Johann Loschmidt. Über den zustand des wärmegleichgewichtes eines systems von körpern. *Wiener Berichte*, 73:128–142, 1876.

[5] Claude E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27(3):379–423, 1948.

[6] John von Neumann. Proof of the quasi-ergodic hypothesis. *Proceedings of the National Academy of Sciences*, 18(1):70–82, 1932.

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DaDoDs
68456@metalurgica.frc.utn.edu.ar

"Scientes ubi inveniatis"